Topic 7 –

Subspaces of $\mathbb{R}^n$

Def: Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ be linearly independent vectors in $\mathbb{R}^n$.

The set of all vectors spanned by $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ is called a subspace of $\mathbb{R}^n$. It is called the span of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ and written

$$\text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r)$$

$$= \left\{ c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c\,\vec{v}_r \;\middle|\; \begin{array}{l} c_1, c_2, \dots, c_r \text{ can be} \\ \text{any real numbers} \end{array} \right\}$$

Let $W = \text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r)$

We say that the dimension of W is r and we write $\dim(W) = r$.

If $\beta = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r]$ then we call β a basis for W.

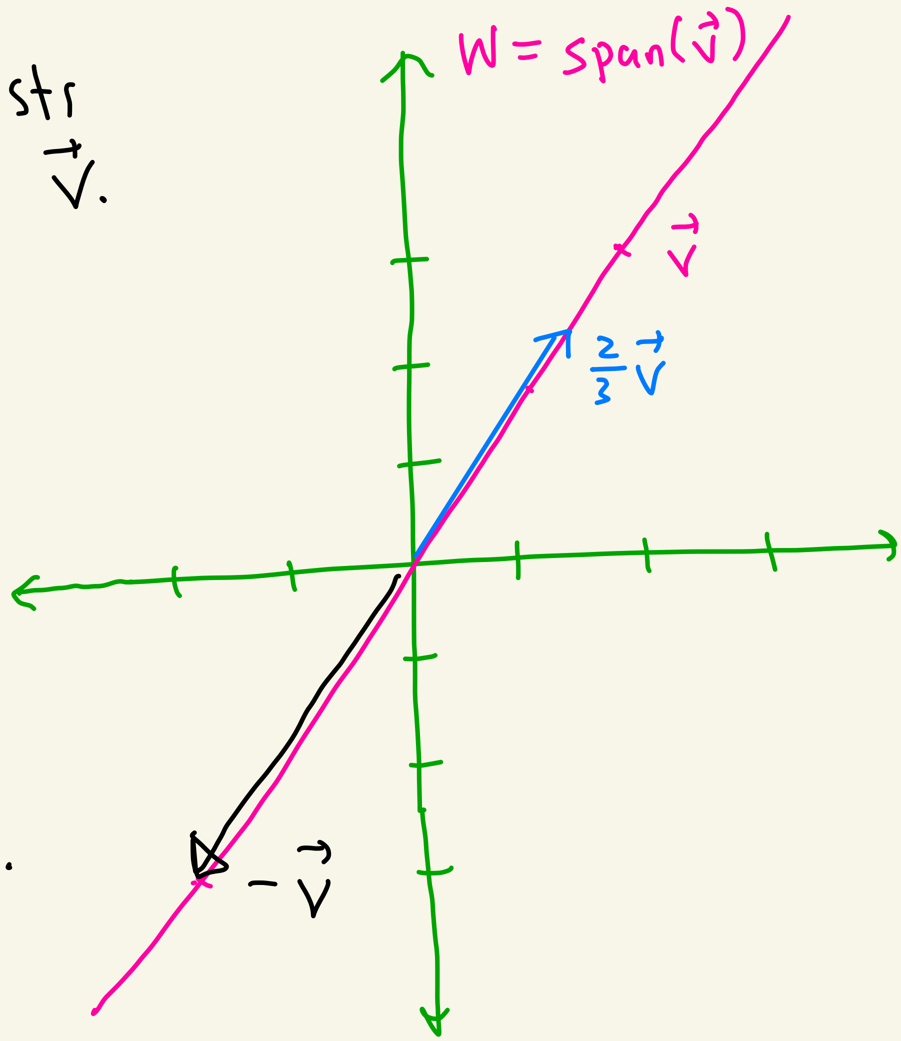
Ex: In $\mathbb{R}^2$ let $\vec{v} = \langle 2,3 \rangle$.

Since $\vec{v}$ is a single non-zero vector, we get that $\{\vec{v}\}$ is a lin. ind. set.

The subspace spanned by $\vec{v}$ is

$$W = \text{span}(\vec{v}) = \{ c_1 \vec{v} \mid c_1 \text{ is any real number} \}$$

This subspace consists of all multiples of $\vec{v}$.



For example,

$1 \cdot \vec{v} = \vec{v} = \langle 2,3 \rangle$

$-1 \cdot \vec{v} = -\vec{v} = \langle -2,-3 \rangle$

$\frac{2}{3}\vec{v} = \langle \frac{4}{3}, 2 \rangle$

are all examples of vectors in W.

Here $\dim(W) = 1$ since $\beta = [\vec{v}]$ is a basis for W with one vector.

Ex: Consider the vector space $\mathbb{R}^2$.

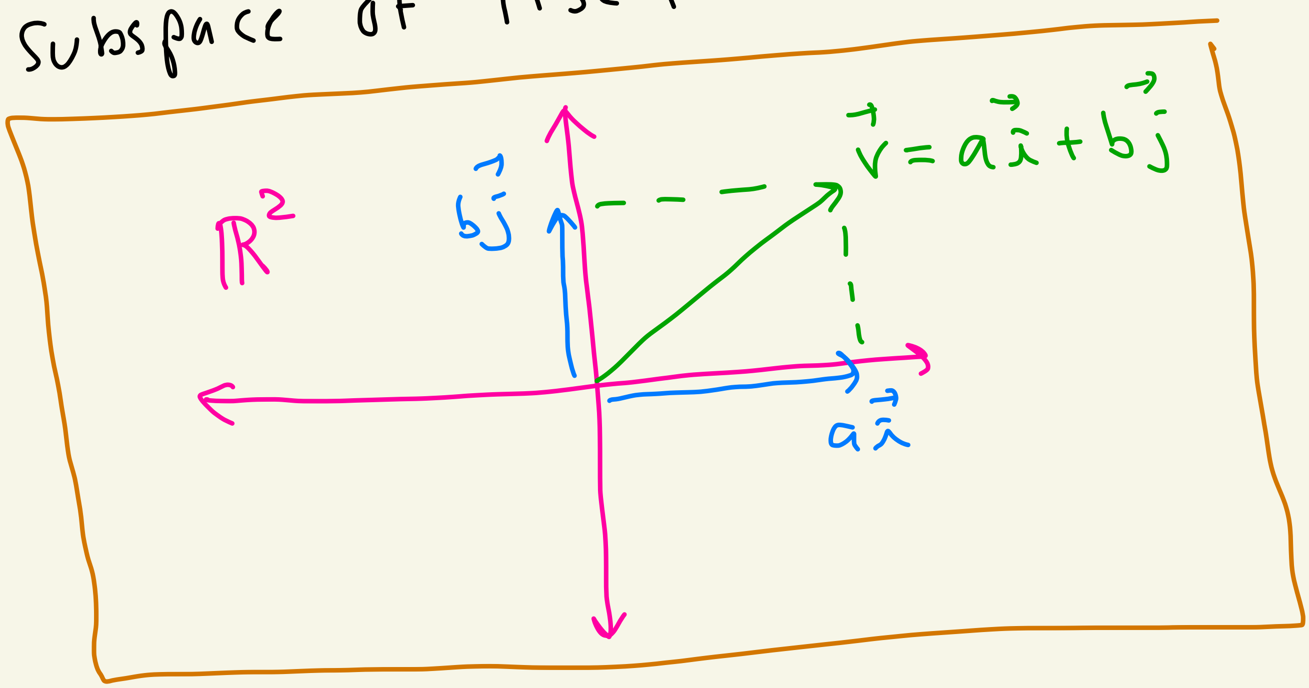
Let $\vec{i} = \langle 1,0 \rangle$, $\vec{j} = \langle 0,1 \rangle$.

We know already that $\beta = [\vec{i}, \vec{j}]$ is a basis for all of $\mathbb{R}^2$ and that every vector $\vec{v} = \langle a,b \rangle$ in $\mathbb{R}^2$ is in $\text{span}(\vec{i}, \vec{j})$ since

$$\begin{aligned}\vec{v} = \langle a,b \rangle &= \langle a,0 \rangle + \langle 0,b \rangle \\ &= a\langle 1,0 \rangle + b\langle 0,1 \rangle \\ &= a\vec{i} + b\vec{j}\end{aligned}$$

Thus, $\text{span}(\vec{i}, \vec{j}) = \mathbb{R}^2$.

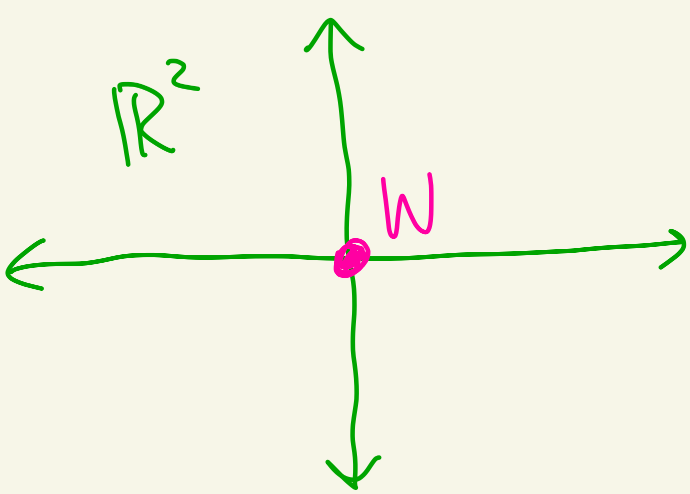
So, $\mathbb{R}^2$ is a 2-dimensional subspace of itself.



There is a special case of subspace that our first def of subspace misses. It is the subspace $\{\vec{0}\}$ that has no basis.

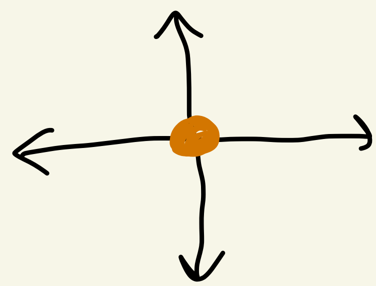
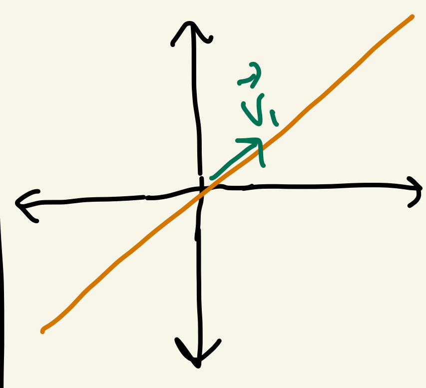
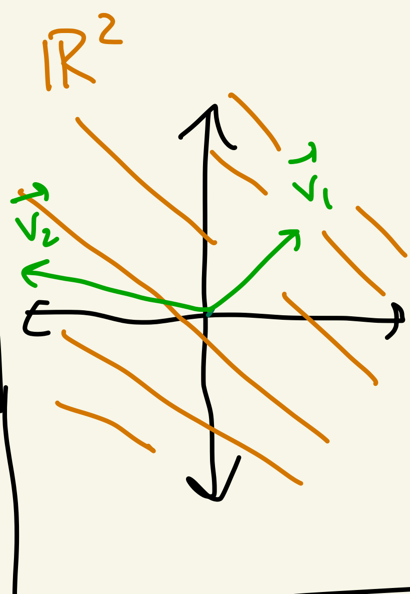
<u>Def:</u> Let $W = \{\vec{0}\}$ in $\mathbb{R}^n$.
Even though W has no basis we define W to be a <u>subspace</u> of $\mathbb{R}^n$. We call W the <u>trivial subspace</u> of $\mathbb{R}^n$. We define the dimension of W to be 0.

<u>Ex:</u> $W = \{\vec{0}\}$ in $\mathbb{R}^2$

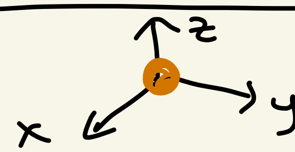
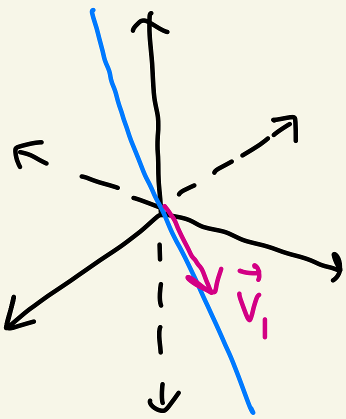
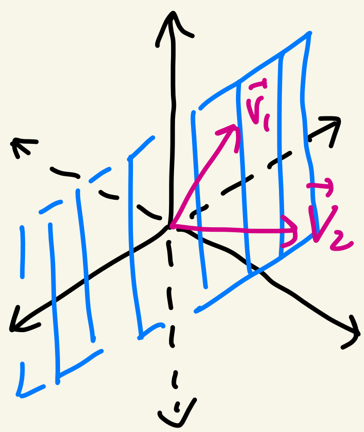
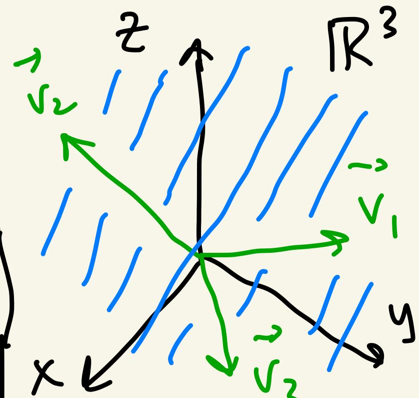


$\dim(W) = 0$

Kinds of subspaces in $\mathbb{R}^2$

dimension m	basis of m lin. ind. vectors	picture of the span of the basis	description
0	there is no basis		$\{\vec{0}\}$
1	$\vec{v}_1$		a line through the origin
2	$\vec{v}_1, \vec{v}_2$		all of $\mathbb{R}^2$

Kinds of subspaces of $\mathbb{R}^3$

dimension m	basis of m lin. ind. vectors	picture of the span of the basis	description
0	there is no basis		$\{\vec{0}\}$
1	$\vec{v}_1$		a line through the origin
2	$\vec{v}_1, \vec{v}_2$		a plane through the origin
3	$\vec{v}_1, \vec{v}_2, \vec{v}_3$		all of $\mathbb{R}^3$

We can't draw a picture of subspaces of $\mathbb{R}^n$ when $n \geq 4$ but it's the same idea. They are "linear" spaces that pass through the origin.

There is another way to make subspaces of $\mathbb{R}^n$.

Homogeneous subspace theorem

Let W consist of all the vectors $\vec{v} = \langle x_1, x_2, \dots, x_n \rangle$ that satisfy a homogeneous system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ \vdots \qquad \qquad \vdots \qquad \qquad \quad \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}$$

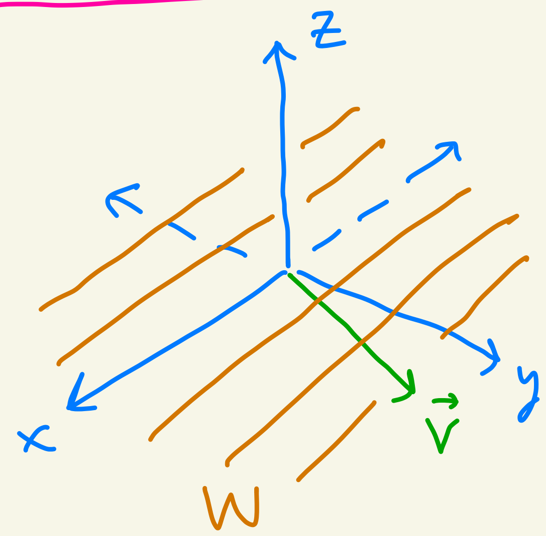
Then W is a subspace of $\mathbb{R}^n$

Conversely, if W is a subspace of $\mathbb{R}^n$ then there exists a homogeneous system of linear equations that W is the solution space to.

Ex: Consider the vector space $\mathbb{R}^3$.

Let $W = \{ \langle x, y, z \rangle \mid z = 0 \}$

By the homogenous subspace theorem, W will be a subspace of $\mathbb{R}^3$.



W consists of all the vectors $\vec{v}$ in the xy-plane

Let's find a basis for W.

Pick a vector

$$\vec{v} = \langle x, y, z \rangle$$

from W.

Then by the def of W, we know $z = 0$

So,

$$\begin{aligned}
\vec{v} &= \langle x, y, 0 \rangle \\
&= \langle x, 0, 0 \rangle + \langle 0, y, 0 \rangle \\
&= x \langle 1, 0, 0 \rangle + y \langle 0, 1, 0 \rangle \\
&= x \vec{i} + y \vec{j}
\end{aligned}$$

So, every vector in W is in the span of two two vectors $\vec{i} = \langle 1, 0, 0 \rangle$ and $\vec{j} = \langle 0, 1, 0 \rangle$.

Let's show that $\vec{i}, \vec{j}$ are lin. ind.

If $c_1\vec{i} + c_2\vec{j} = \vec{0}$ then

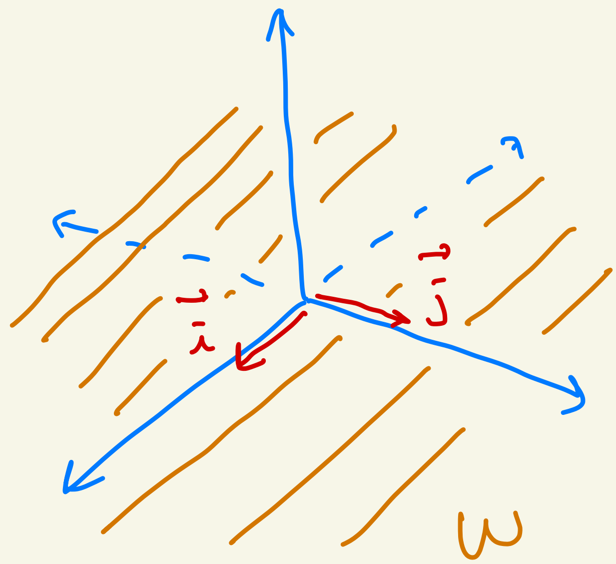
$$c_1\langle 1,0,0\rangle + c_2\langle 0,1,0\rangle = \langle 0,0,0\rangle$$

Thus,

$$\langle c_1, c_2, 0\rangle = \langle 0,0,0\rangle$$

So, $c_1 = 0$, $c_2 = 0$.

Since the only solution to $c_1\vec{i} + c_2\vec{j} = \vec{0}$ is $c_1 = 0, c_2 = 0$, we know that $\vec{i}, \vec{j}$ are lin. ind.



Thus, $\beta = [\vec{i}, \vec{j}]$ is a basis for W. So, W is a subspace of $\mathbb{R}^3$ of dimension 2.

W is the xy-plane in 3 dimensions.

Ex: Show that

$$W = \left\{ \langle x, y, z \rangle \;\middle|\; \begin{matrix} x + y = 0 \\ y - 5z = 0 \end{matrix} \right\}$$

is a subspace of $\mathbb{R}^3$, find a basis for W, and the dimension of W.

By the homogeneous subspace theorem, W is a subspace of $\mathbb{R}^3$. Lets find a basis for W.

Suppose $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in W.

Then:

$$\boxed{\begin{aligned} x + y &= 0 \quad ① \\ y - 5z &= 0 \quad ② \end{aligned}}$$

This system is already reduced. We have:

$$\boxed{\begin{aligned} x &= -y \quad ① \\ y &= 5z \quad ② \\ z &= t \quad ③ \end{aligned}}$$

So,

$z = t$

$y = 5z = 5t$

$x = -y = -5t$

Thus, if $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in W then

$$\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5t \\ 5t \\ t \end{pmatrix} = t \begin{pmatrix} -5 \\ 5 \\ 1 \end{pmatrix}$$

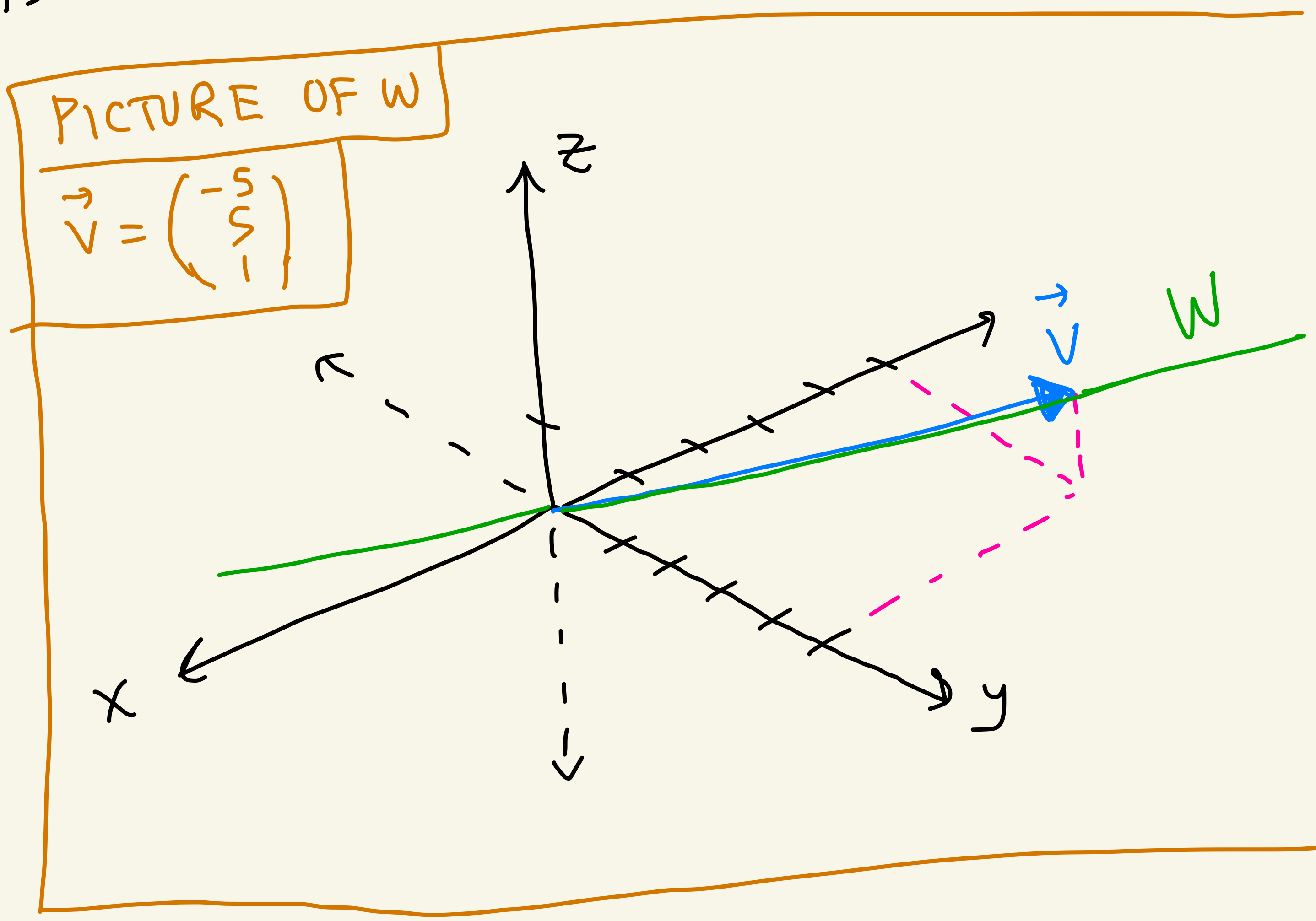
So, W is spanned by the vector $\begin{pmatrix} -5 \\ 5 \\ 1 \end{pmatrix}$

Since $\left\{ \begin{pmatrix} -5 \\ 5 \\ 1 \end{pmatrix} \right\}$ consists of a single non-zero vector we know it forms a lin. ind. set.

Thus, $\beta = \left[\begin{pmatrix} -5 \\ 5 \\ 1 \end{pmatrix} \right]$ is a basis for W.

And $\dim(W) = 1$.

W is a line in $\mathbb{R}^3$.



<u>Theorem:</u> The dimension of a subspace W of $\mathbb{R}^n$ is well-defined. That is, if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_a$ are linearly independent with $\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_a) = W$ and $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_b$ are linearly independent with $\text{span}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_b) = W$, then $a = b$.

<u>Theorem:</u> W is a subspace of $\mathbb{R}^n$ if and only if

1. $\vec{0}$ is in W
2. If $\vec{w}_1, \vec{w}_2$ are in W, then $\vec{w}_1 + \vec{w}_2$ is in W.
3. If $\vec{w}$ is in W and α is a scalar in $\mathbb{R}$, then $\alpha\vec{w}$ is in W.